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Tree–connected peer group situations and peer group games

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Abstract

A class of cooperative games is introduced which arises from situations in which a set of agents is hierarchically structured and where potential individual economic abilities interfere with the behavioristic rules induced by the organization structure. These games form a cone generated by a specific class of unanimity games, namely those based on coalitions called peer groups. Different economic situations like auctions, communication situations, sequencing situations and flow situations are related to peer group games. For peer group games classical solution concepts have nice properties.

JEL classification: C71.
Keywords: cooperative game, peer group game, graph–restricted game, auction, sequencing, airport game.

1 Introduction

There are many economic situations where the social configuration of the organization influences the potential economic possibilities of all the groups of agents. Several authors have used a game theoretical approach for analysing

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the consequences of this kind of constraints on economic cooperative behavior. They have separately considered the potential individual economic possibilities as described by a cooperative game with transferable utility (TU-game) and the structure of the agent set induced by the social configuration of the organization, and then modified the game accordingly. We mention cooperative games with arbitrary communication structures (cf. Myerson (1977, 1980), Owen (1986), Borm et al. (1994)) and games with permission structures (cf. Gilles et al. (1992)).

In this paper we introduce a class of cooperative games naturally arising from situations in which the set of agents is (strict) hierarchically structured and where the potential individual economic possibilities interfere with the behavioristical rules induced by the organization structure. We are thinking of certain auctions, sequencing situations, flow situations and communication situations of a special type. Every agent in a strict hierarchy has a relationship with the leader either directly or indirectly with the help of one or more other agents. The economic possibilities of an agent are restricted by his position in the hierarchy. The important group for an agent in such a situation is that consisting of the leader, the agent himself and all the intermediate agents that exist in the given hierarchy between the agent and the leader, because only by this cooperation the agent's potential economic possibilities can become effective. We call such a group of agents a *peer group*. Our game theoretical approach is based on peer groups of agents and on an integrative view of the economic possibilities and the organization structure.

Recall that a TU-game is a pair $< N, v >$, where $N = \{1, 2, ..., n\}$ is a finite set of players and $v : 2^N \to \mathbb{R}$ is a characteristic function, that assigns to each coalition $S \subset N$ the worth (reward) $v(S)$, with $v(\emptyset) = 0$.

The agents' hierarchy is described by a rooted directed tree with the leader located in the root and each other agent in a distinct node. This tree uniquely determines the peer group structure: each agent's peer group corresponds to the agents in the unique path connecting the agent's node with the root in the tree.

Tree–connected peer group situations are introduced as triplets consisting of the set of agents involved, the peer group structure describing the organization's social configuration and a real–valued vector that gives the potential individual economic possibilities of the agents.

To each tree–connected peer group situation we associate a TU–game, which we call a *peer group game*, with the agents as players and the characteristic function defined for each coalition by pooling the individual economic possibilities of those members with the corresponding peer groups inside the coalition. Thus, the peer groups are essentially the only coalitions that can generate a non–zero payoff within a peer group game.

Our main result states that the family of peer group games forms a cone generated by a specific class of unanimity games, namely those based on coalitions called peer groups. With the use of this result we show that the cone of peer group games lies in the intersection of the cones of convex games and monotonic veto-rich games with the leader as veto-player. As a result classical solution concepts for peer group games have nice properties. We study solutions like: the core (cf. Gillies (1953)), the bargaining set (cf. Aumann and Maschler (1964)), the kernel (cf. Davis and Maschler (1965)), the nucleolus (cf. Schmeidler (1969), Kohlberg (1971)), the Weber set (cf. Weber (1988)), the Shapley value (cf. Shapley (1953)), the τ–value (cf. Tijs (1981)), the selectope (cf. Hammer et al. (1977)). Special attention is paid to the nucleolus of peer group games corresponding to line–graph connected peer groups. We characterize the nucleolus of line–graph peer group games as the unique solution of n equations.

The content of the paper is organized as follows. In the next section we introduce our model, give the necessary definitions and prove our main result. Some economic situations related to peer group games are described in section 3. In section 4 properties and relations of solutions for peer group games are discussed. Section 5 deals with the nucleolus of line–graph peer group games.

2 Peer group situations and games

Let $N = \{1, 2, ..., n\}$ be a finite set of agents with social as well as individual economic characteristics.

The social features are given by a strict hierarchy defining the agents' relationships. Such a hierarchy can be described by a rooted directed tree[1] T with $N = \{1, 2, ..., n\}$ as node set, agent 1 (the leader) as root, and each other agent located in a node. Here is an example of such a tree representation where five agents are involved.

[1]By a *rooted directed tree* we mean a directed graph with one distinguished node as root in which for each node there is a unique directed path from the root to that node.

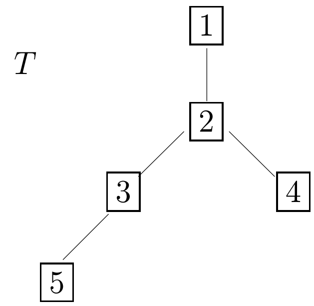

Figure 2.1.

Chain–like hierarchies will be represented by line–graphs[2] or chains.

The individual features are agents' potential economic possibilities, described by a vector $a \in \mathbb{R}_+^N$, where a_i is the gain which can be generated by agent i if all his superiors cooperate with him.

We model the social constraints in economic behavior by means of T–connected peer groups of agents.

Definition 2.1. For each agent i, $i{\in}N$, we call the *peer group of agent* i the subset consisting of all the agents $[1,i]$ in the path of T connecting 1 to i.

Example 2.1. The set of all the peer groups corresponding to the tree T in Figure 2.1 is

$$[1,1]=\{1\},\ [1,2]=\{1,2\},\ [1,3]=\{1,2,3\},$$
$$[1,4]=\{1,2,4\},\ [1,5]=\{1,2,3,5\}.$$

Remark 2.1. It is the peer group of agent i that imposes social constraints in agent i's economic behavior. Agent i can only become "effective" if he is in cooperation with all the other members of his peer group.

Definition 2.2. A *peer group structure on* N *induced by* T is a mapping P which associates to each agent i in N the peer group of agent i

$$P : N \longrightarrow 2^N, \text{ where } P(i)=[1,i].$$

Remark 2.2. $P(i)=\{j \in N \mid j \underset{T}{\preceq} i\}$, where $j \underset{T}{\preceq} i$ means that j lies on the path from the root 1 to i.

[2]By a *line–graph* or *chain* we mean a tree whose nodes are located on a single path.

Definition 2.3. A T–*connected peer group situation* is a triplet $< N, P, a >$, where N is the set of agents involved, P is the peer group structure on N induced by T, and $a \in \mathbb{R}_+^N$ is the vector describing the individual potential economic possibilities.

To each T–connected peer group situation (*pg-situation*) we associate a TU cooperative game called in the following *peer group game* (*pg–game*).

Definition 2.4. Given a T–connected peer group situation $< N, P, a >$ we call the corresponding *peer group game* the TU–game $< N, v_{P,a} >$, or shortly $< N, v >$, with $N = \{1, 2, ..., n\}$ and

$$v(S) = \sum_{i:P(i) \subset S} a_i, \ \forall S \subset N; \ v(\emptyset) = 0.$$

Example 2.2. Let $< N, P, a >$ be the T–connected peer group situation with $N = \{1, 2, 3, 4, 5\}$, P the peer group structure on N induced by the tree T in Figure 2.1, and $a \in \mathbb{R}_+^N$. Then the corresponding peer group game is given by

$$\begin{aligned}
&v(\{1\}) = v(\{1,3\}) = v(\{1,4\}) = v(\{1,5\}) = v(\{1,3,4\}) = \\
&\qquad = v(\{1,3,5\}) = v(\{1,4,5\}) = v(\{1,3,4,5\}) = a_1; \\
&v(\{1,2\}) = v(\{1,2,5\}) = a_1 + a_2; \\
&v(\{1,2,3\}) = a_1 + a_2 + a_3; \\
&v(\{1,2,4\}) = v(\{1,2,4,5\}) = a_1 + a_2 + a_4; \\
&v(\{1,2,3,4\}) = a_1 + a_2 + a_3 + a_4; \\
&v(\{1,2,3,5\}) = a_1 + a_2 + a_3 + a_5; \\
&v(N) = a_1 + a_2 + a_3 + a_4 + a_5; \ v(S) = 0 \text{ otherwise.}
\end{aligned}$$

Remark 2.3. If $1 \notin S$ then $v(S) = 0$.

The peer groups are essentially the only payoff generating coalitions within a peer group game. Each peer group game can be expressed as a nonnegative combination of the unanimity games corresponding to the peer groups.

Let $u_{[1,i]}$ be the unanimity game corresponding to the peer group $[1, i]$ of agent i. Then

$$v = \sum_{i=1}^{n} a_i u_{[1,i]} \tag{2.1}$$

So, in fact, a_i is the Harsanyi dividend of the peer group $[1,i]$.

Let $\alpha,\beta\geq 0$, $a,b\in\mathbb{R}^N_+$, and let $<N,P,a>$, $<N,P,b>$, $<N,P,\alpha a+\beta b>$ be peer group situations corresponding to the hierarchy T described by P. Then for the corresponding peer group games $v_{P,a}$, $v_{P,b}$, $v_{P,\alpha a+\beta b}$ we have

$$v_{P,\alpha a+\beta b}=\alpha v_{P,a}+\beta v_{P,b}.$$

So, peer group games form a cone $\{<N,v_{P,a}>\mid a\in\mathbb{R}^N_+\}$, which is generated by the independent subset $\{u_{[1,i]}\mid i\in N\}$ of unanimity games corresponding to peer groups.

Note that from (2.1) it follows that the cone of peer group games is a subcone in the cone of convex games because unanimity games are convex (Shapley (1971)) and peer group games are nonnegative combinations of convex games.

Note also that from (2.1) it follows that peer group games are monotonic and that agent 1 (the leader) is a veto player. So, the cone of peer group games is also a subcone in the cone of monotonic games with 1 as veto player.

Peer group games are also superadditive. Then $w=v-a_1u_{[1,1]}$ is zero–normalized and superadditive. We study its relation with T–component additive games[3].

Recall that according to Potters and Reijnierse (1995) a game $<N,v>$ is called a T–component additive game if $<N,v>$ is a superadditive zero–normalized game with $R_T(v)=v$, where T is a tree and

$$R_T(v)(S)=\sum_{U\in S/T}v(U),$$

where S/T is the set of connected components of S in T. For each $i\in N\backslash\{1\}$, $u_{[1,i]}$ is a T–component additive game because

$$R_T(u_{[1,i]}(S))=\sum_{U\in S/T}u_{[1,i]}(U)=\begin{cases}1, & [1,i]\subset S\\ 0, & \text{otherwise.}\end{cases}$$

So, $R_T(u_{[1,i]})=u_{[1,i]}$. Then

$$R_T(w)=\sum_{i\in N\backslash\{1\}}a_iR_T(u_{[1,i]})=\sum_{i\in N\backslash\{1\}}a_iu_{[1,i]}=w.$$

[3] *T–component additive games* are introduced in Potters and Reijnierse (1995) as Γ–component additive games.

This means that $v - a_1 u_{[1,1]}$ is an element in the cone of T–component additive games with $N \setminus \{1\}$ as player set.

So, we proved the following theorem.

Theorem 2.1.

(i) *The peer group games corresponding to the T–connected peer group situations $< N, P, a >$ with N, P fixed and $a \in \mathbb{R}_+^N$ form a cone $PGG(N, P)$;*

(ii) *The cone $PGG(N, P)$ is in the intersection of the cones of convex games and monotonic veto rich games with 1 as veto player;*

(iii) *For each peer group game v, the zero–normalization $v - a_1 u_{[1,1]}$ is an element in the cone of T–component additive games.*

Remark 2.4. In relation with Theorem 2.1 the following question could arise: is each game v in the intersection of convex games and monotonic veto–rich games with 1 as veto player and satisfying the condition $v - v(\{1\})u_{[1,1]}$ in the cone of T–component additive game a peer group game? The answer is negative. Consider a T–component additive game with

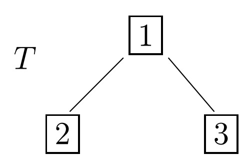

Figure 2.2.

given by

$$v = 5u_{\{1,2\}} + 7u_{\{1,3\}} + 10u_{\{1,2,3\}}.$$

Note that v is a convex game, a T–component additive game, and also a monotonic game with 1 as veto player, but it is not a peer group game because $\{1, 2, 3\}$ is not a peer group.

3 Economic situations related to peer group situations

In this section auction situations, communication situations and sequencing situations are considered, leading to *pg*–situations and *pg*-games. To a peer group situation one can also assign a max flow situation such that the corresponding peer group game and flow game coincide. Further a formal mathematical relation between airport problems and peer group situations is discussed.

3.1 Auction situations and peer group situations

Let us concentrate in this subsection on sealed bid second price auction situations (cf. Rasmusen (1989), chapter 11). In such a situation there is a seller of an object who has a reservation price, say r, which is the lowest price for which he wants to sell the object and which is known to the potential bidders. We suppose that there are n bidders (players) $1, 2, ..., n$, each of them submitting one bid $b_1, b_2, ..., b_n$ in an envelope. After opening the envelopes, the bidder with the highest bid obtains the object at the price of the second highest bid.

Let w_i be the value for player i of the object, which is not necessarily known by the other players. Suppose

$$w_1 > w_2 > w_3 > \cdots > w_n \geq r. \tag{3.1}$$

Note that if player i acts alone, it is optimal for him (a dominant strategy) to bid his own value, so $b_i = w_i$. This leads to a payoff $v(\{i\}) = 0$ if $i \neq 1$ and to a payoff $v(\{1\}) = w_1 - w_2$ for player 1, because player 1 obtains the object for a price w_2. If all players in $N = \{1, 2, ..., n\}$ decide to cooperate, a dominant strategy is that player 1 bids $b_1 = w_1$ and the others r, so $b_i = r$ if $i \in N \setminus \{1\}$. Then the object goes to player 1 at price r and the payoff to N is $v(N) = w_1 - r$. If a coalition $S \neq N$ works together (secretly or not), they detect which player in S has the highest value. If this is player $i(S)$, then he bids $w_{i(S)}$ and the others bid r. This is a dominant bidding strategy for S. Supposing that the other players (or groups) in $N \setminus S$ play also their dominant bidding strategy we can consider two cases.

Case 1. Player 1 is not in S, so $i(S) \neq 1$. Then the object goes to player 1 (or the group to which 1 belongs) in $N \setminus S$. The value $v(S)$ of S is in this case 0.

Case 2. Player 1 is in S. Then the highest bid is w_1 and the second highest bid w_{k+1} if $[1,k] \subset S$ and $k+1 \notin S$. (Remember that $b_i = r$ for $i = 2, ..., k$). In this case the value of the coalition S is $v(S) = w_1 - w_{k+1}$, because player 1 obtains the object at price w_{k+1}. For the coalitional game $< N, v >$ corresponding to this auction situation we have

Proposition 3.1. $< N, v >$ *coincides with the peer group game corresponding to the T-connected peer group situation* $< N, P, a >$ *where* $N = \{1, 2, ..., n\}$, *T is the line–graph with root* 1 *and with arcs* $(1,2), (2,3), ...,$ $(n-1, n)$ *and where* $a_i = w_i - w_{i+1}$ *for* $i \in N$ *with* $w_{n+1} = r$.

Proof. Let $< N, v' >$ be the peer group game, corresponding to the peer group situation described in the theorem. Then $v' = \sum_{i=1}^{n} (w_i - w_{i+1}) u_{[1,i]}$. So $v'(S) = 0$ if $1 \notin S$ and $v'(S) = \sum_{i=1}^{k} (w_i - w_{i+1}) = w_1 - w_{k+1}$ if $[1,k] \subset S$ and $k+1 \notin S$. Hence, $v = v'$. ■

Example 3.1. Suppose that in a sealed bid second price auction there are three bidders with values for the object of 100, 80, 50, respectively, and suppose that the reservation price is 25. Then the corresponding *pg*–game equals $20u_{\{1\}} + 30u_{\{1,2\}} + 25u_{\{1,2,3\}}$. If all bidders work together, bidder 1 bids 100, bidders 2 and 3 bid 25, so the object goes to player 1 who pays 25.

Remark 3.1. Consider a first price sealed bid auction for which the values $w_1, w_2, ..., w_n$ and the reservation price r (denoted also by w_{n+1}) are common knowledge among the bidding agents and where also (3.1) holds. Suppose the minimal increment is ε and $\varepsilon < w_i - w_{i+1}$ for all $i \in N$. Then it is optimal for player i to submit a bid $w'_{i+1} := w_{i+1} + \varepsilon$. For a subgroup S with $[1, i] \subset S$ and $i + 1 \notin S$ it is 'optimal' that player 1 bids $w_{i+1} + \varepsilon$ and the other players in S bid r. The corresponding cooperative game $< N, v >$ is a peer group game, because

$$v = \sum_{i=1}^{n} (w'_i - w'_{i+1}) u_{[1,i]} \quad \text{where } w'_1 = w_1.$$

3.2 Graph–restricted binary communication situations

Let $< N, P, a >$ be as before a T-connected peer group situation. Consider situations where gains are made via binary interactions (communications) of a central agent 1 with each of the other agents $i \in N$, resulting in a gain a_i, but where communication restrictions hold, described by the tree T. One can think of binary interactions of different kind such as information exchange between 1 and i, or import (export) of goods via harbour 1 for agent i, or approval by 1 of a planned action of player i. If there were no communication restrictions, the corresponding game $< N, w >$ should be given by $w = \sum_{i=1}^{n} a_i u_{\{1,i\}}$. With the communication restriction given by the tree T, we obtain the graph–restricted game (cf. Myerson (1977)) $v = w_{|T}$, where v turns out to be the peer group game $v = \sum_{i=1}^{n} a_i u_{[1,i]}$ corresponding to $< N, P, a >$.

3.3 Sequencing situations

Recall that a sequencing situation (cf. Curiel et al. (1989)) is a triplet (σ_0, p, α), where σ_0 is the initial order, $p = (p_i)_{i \in N}$ with $p_i > 0$ is the processing time of customer i, and $\alpha = (\alpha_i)_{i \in N}$, where α_i is the cost per unit of time for i. The urgency index of i is given by $u_i = p_i^{-1} \alpha_i$. It is well known (cf. Smith (1956)) that it is optimal to serve the agents according to their urgency, the most urgent first etc., and this order can be obtained by neighbour switches. The corresponding cost savings game is a nonnegative combination of unanimity games on neighbours that switch

$$v = \sum_{(k,\ell), k<\ell} g_{k,\ell} u_{[k,\ell]}, \text{ where } g_{k,\ell} = (p_k \alpha_\ell - p_\ell \alpha_k)_+, \text{ i.e. } g_{k,\ell} = \max\{0, p_k \alpha_\ell - p_\ell \alpha_k\}.$$

Some sequencing situations also lead to peer group games. We are thinking of special sequencing situations in which the initial order $\sigma_0 = (1, 2, ..., n)$ of n customers is such that the following relation between urgency indices holds

$$u_2 > u_3 > \cdots > u_n. \tag{3.2}$$

In the case that (3.2) holds, the optimal order is obtained only by neighbour switches between 1 and some other customers; this means that all $g_{k,\ell}$

with $k \neq 1$ are zero. So, the considered sequencing situations lead to peer group games of the form

$$v = \sum_{i=2}^{n} g_{1,i} u_{[1,i]}, \text{ where } g_{1,i} = (p_1\alpha_i - p_i\alpha_1)_+.$$

Such a peer group game corresponds to a T–connected peer group situation $< N, P, a >$, where T is the line–graph with arcs $(1,2), (2,3), ..., (n-1, n)$, corresponding to the initial order in the sequencing situation, and where $a_i = g_{1,i}$ for each $i \in N \setminus \{1\}$ and $a_1 = 0$.

3.4 Flow situations

Peer group games can also arise from flow situations. Let us consider the tree T from Figure 2.1 that generates the peer group game given in Example 2.2, and construct the corresponding flow situation. We have to add two special nodes called the source and the sink and the following arcs: the arc connecting the source with agent 1 with infinite capacity, and for each $i \in N$ the arc with capacity a_i connecting agent i with the sink. The following flow situation results, where the notation is owner/capacity.

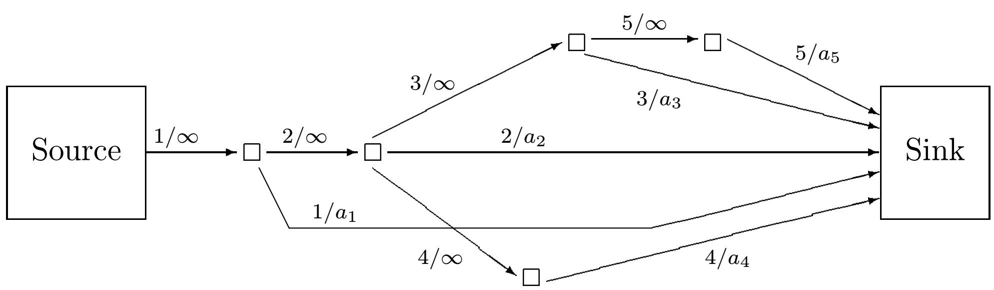

Figure 3.1.

The corresponding flow game (cf. Kalai and Zemel (1982)) coincides with the peer group game.

3.5 Airport games and peer group games

Landing fee problems for planes of different types have generated an interesting class of cooperative games, namely airport games (cf. Littlechild and Owen (1977)). Suppose planes of players $1, 2, ..., n$ need landing strips of length $\ell_1, \ell_2, ..., \ell_n$ with $\ell_1 > \ell_2 > \cdots > \ell_n$ and related costs $c_1 > c_2 > \cdots > c_n$. The corresponding airport game $< N, c >$ is given by

$$c = c_n u_N^* + (c_{n-1} - c_n) u_{N \setminus \{n\}}^* + \cdots + (c_1 - c_2) u_{\{1\}}^*,$$

where $< N, u_S^* >$ is a game with $u_S^*(T) = 1$ if $S \cap T \neq \emptyset$ and $u_S^*(T) = 0$ otherwise. The Shapley value of this game gives an interesting and appealing way to solve the landing fee problems. The dual game $< N, c^* >$ corresponding to $< N, c >$ is given by $c^*(S) = c(N) - c(N \setminus S)$ for each $S \subset N$. Note that $c^* = c_n u_N + \sum_{i=1}^{n-1} (c_i - c_{i+1}) u_{\{1,2,...,i\}}$, which is clearly a peer group game. We did not yet exploit the duality relation between airport games and line–graph peer group games.

4 Solutions for peer group games

By Theorem 2.1 (ii), peer group games are convex games and veto rich games. This fact implies many nice properties and relations for solutions.

Convex games were introduced in Shapley [1971] where it is proved that they have nonempty cores and the "regular" structure of the core is studied. Additional nice properties for solutions of convex games are proved in Maschler, Peleg and Shapley (1972), Driessen (1988), Curiel (1997). Thus it is shown that for convex games some solution concepts (the bargaining set, the Weber set) coincide with the core, and other solutions (the Shapley value, the kernel, the nucleolus) occupy central positions in the core.

Monotonic veto–rich games were introduced by Arin and Feltkamp (1997) where special properties for the kernel and nucleolus are proved, and an efficient algorithm for computing the nucleolus is given.

Peer group games are also positive games, i.e. nonnegative combinations of unanimity games (see (2.1)); other results concerning solutions follow.

Theorem 4.1. *For peer group games the following properties of solution concepts*[4] *hold:*

(i) *The bargaining set $\mathcal{M}(v)$ coincides with the core $C(v)$;*

(ii) *The kernel $\mathcal{K}(v)$ coincides with the pre–kernel $\mathcal{K}^*(v)$ and the pre-kernel consists of a unique point which is the nucleolus of the game;*

[4] We refer to solutions for the grand coalition.

(iii) *The nucleolus* $Nu(v)$ *occupies a central position in the core and is the unique point satisfying*

$$\begin{array}{l} Nu(v) = \{x \in C(v) \mid s_{ij}(x) = s_{ji}(x),\ \forall i,j\},\ \textit{where} \\ s_{ij}(x) = \max\{v(S) - x(S) \mid i \in S \subset N \setminus \{j\}\}; \end{array}$$

(iv) *The core* $C(v)$ *coincides with the Weber set* $W(v)$, *that is* $C(v) = \mathrm{conv}\{m^\sigma(v) \mid \sigma$ *is a permutation of the players*$\}$ *and* $m^\sigma(v)$ *is the marginal vector w.r.t.* σ*;*

(v) *The Shapley value* $\Phi(v)$ *is the center of gravity of the extreme points of the core and is given by*

$$\Phi_i(v) = \sum_{j: i\in P(j)} \frac{a_j}{|P(j)|},\ i \in N,$$

where $P(j)$ *is the peer group of player* j *and* $|P(j)|$ *means the number of elements in* $P(j)$*;*

(vi) *The* τ*-value is given by*

$$\tau(v) = \alpha(a_1, 0, ..., 0) + (1-\alpha)(M_1(v), M_2(v), ..., M_n(v)\},$$

where $M_i(v) = \sum_{j: i\in P(j)} a_j$ *and* $\alpha \in [0,1]$ *is such that* $\sum_{i=1}^{n} \tau_i(v) = v(N)$.

(vii) *The core* $C(v)$ *coincides with the selectope* $S(v)$ *of the game, where* $S(v) := \mathrm{conv}\{m^\beta(v) \in \mathbb{R}^N | \beta : 2^N \setminus \{\emptyset\} \to N$ *with* $\beta(S) \in S\}$, *where* $m^\beta(v)$ *is the selector value corresponding to* β*;*

(viii) *There exist population monotonic allocation schemes (pmas).*

Proof. (i), (ii) and (iii) follow from convexity and are shown in Maschler, Peleg and Shapley (1972) and Maschler (1984).

(iv) is proved in Driessen (1988) and Curiel (1997).

(v) From convexity it follows that the Shapley value is in the barycenter of the core. According to the definition of the Shapley value (cf. Shapley (1953)) the players in each peer group split equally the Harsanyi dividend of their peer group. The expression for $\Phi(v)$ results then from (2.1).

(vi) The τ-value for peer group games can easily been calculated, because convexity implies semiconvexity (cf. Driessen (1988)), so

$$\tau_i(v) = \alpha v(\{i\}) + (1-\alpha)(v(N) - v(N \setminus \{i\})),\ i \in N,$$

where α is such that efficiency holds.

The expression for $M_i(v) = v(N) - v(N \setminus \{i\})$ follows from (2.1) and linearity.

(vii) From (2.1) it follows that peer group games are positive games, so according to Theorem 2 in Derks et al. (2000) the core and selectope coincide. The selectope consists of all possible reasonable ways to distribute the dividends of peer groups among the players.

(viii) is also a consequence of convexity. Sprumont (1990) shows that each element of the core of a convex game is extendable to a pmas.

5 The nucleolus for line–graph peer group games

The purpose of this section is to prove that the nucleolus of a *pg*-game corresponding to a line–graph is the unique solution of n equations. We exploit the fact that the nucleolus is a core element and that it is (the unique element) in the prekernel.

In the following lemma and theorem $< N, v >$ is the *pg*-game corresponding to $< N, P, a >$ where $P = \{\{1\}, \{1,2\}, \{1,2,3\}, ..., \{1,2,...,n\}\}$, $a \in \mathbb{R}^n_+$, and z is the nucleolus of $< N, v >$.

Lemma 5.1. *For each* $i \in \{1, ..., n-1\}$ *the nucleolus satisfies the equation* $z_{i+1} = \min\{z_i, Z_i - A_i\}$*, where* $Z_i = \sum_{k=1}^{i} z_k,\ A_i = \sum_{k=1}^{i} a_k$.

Proof. Since z is a core element we have

(5.1) $$z_1 \geq a_1 \geq 0,\ z_k \geq 0 \text{ for } k \in \{2, ..., n\};$$

(5.2) $$z(T) \geq v(T) \text{ for all } T \in 2^N.$$

Since z is a prekernel element we have

(5.3) $$s_{i,i+1}(z) = s_{i+1,i}(z) \text{ for each } i \in \{1, 2, ..., n-1\}.$$

In view of (5.3) it is sufficient to prove for $i \in \{1, ..., n-1\}$

$$s_{i+1,i}(z) = -z_{i+1}, \tag{5.4}$$

$$s_{i,i+1}(z) = \max\{-z_i, A_i - Z_i\}. \tag{5.5}$$

To prove (5.4), take a coalition S with $i+1 \in S$ and $i \notin S$. Let U be the largest peer group in S if there is one; otherwise, let $U = \emptyset$. Then

$$\begin{gathered} e(S,z) = v(S) - z(S) = v(U) - z(S) \leq \\ \leq v(U) - z(U) - z_{i+1} \leq -z_{i+1} = e(\{i+1\}, z), \end{gathered}$$

where the first inequality follows from (5.1) and the second from (5.2) with U in the role of T. So,

$$s_{i+1,i}(z) = \max\{e(S,z) \mid i \notin S, i+1 \in S\} = e(\{i+1\}, z) = -z_{i+1}.$$

To prove (5.5), take a coalition S with $i \in S$, $i{+}1 \notin S$. Then $S = S_1 \cup \{i\} \cup S_2$, where $S_1 = S \cap [1, i-1]$ and $S_2 = S \cap \{i+2, ..., n\}$. For $i = 1$ we interpret $[1, i-1]$ as the empty set $\emptyset$ and $[i+2, n] = \emptyset$ if $i = n-1$ or $i = n$.

Let T_1 be the largest peer group in S_1, if there are peer groups in S_1; otherwise, let $T_1 = \emptyset$.

We consider two cases: $T_1 = [1, i-1]$, $T_1 \neq [1, i-1]$. Note that for $i = 1$ we have only the first case.

Case 1. Let $T_1 = [1, i-1]$. Then

$$e(S,z) = v(S) - z(S) = v([1,i]) - z(S) \leq v([1,i]) - \sum_{k=1}^{i} z_k = e([1,i], z),$$

where the inequality follows from (5.1).

For $i{=}1$ one obtains $s_{1,2}(z){=}\max\{e(S,z)|1{\in}S, 2{\notin}S\}{=}e([1,1], z)$ $= v(\{1\}) - z_1 = a_1 - z_1 = \max\{-z_1, a_1 - z_1\}$. So, (5.5) holds for $i = 1$.

Case 2. Let T_1 be a proper subset of $[1, i-1]$, where $i > 1$. Then

$$e(S,z) = v(S) - z(S) = v(T_1) - z(S) \leq v(T_1) - z(T_1) - z_i \leq -z_i = e(\{i\}, z),$$

where the inequalities follow from (5.1) and (5.2), respectively.

Hence, for $i \geq 2$ we have

$$\begin{aligned} s_{i,i+1}(z) &= \max\{v(S) - z(S) \mid i \in S,\ i+1 \notin S\} = \\ &= \max\{e([1,i],z), e(\{i\},z)\} = \max\{A_i - Z_i, -z_i\}. \end{aligned}$$

■

Theorem 5.1. *The nucleolus z of $< N, v >$ is the unique solution of the n equations*

$$Z_n = A_n, z_i = \min\{z_{i-1}, Z_{i-1} - A_{i-1}\},\ i = 2, ..., n.$$

Proof. Let $M = \{x \in \mathbb{R}^n | x_{i+1} = \min\{x_i, X_i - A_i\}$, for each $i \in \{1, 2, ..., n-1\}\}$. Note that for $x \in M$, the first coordinate uniquely determines the other coordinates $x_2, x_3, ..., x_n$, i.e. x_1 determines x_2, then x_3 is uniquely determined by x_1 and x_2, and so on. Note also that for $x, x' \in M$, if $x_1 > x'_1$, then $x_k > x'_k$ for each $k \in \{2, ..., n\}$, so $\sum_{i=1}^{n} x_i > \sum_{i=1}^{n} x'_i$. This implies that there is at most one element in M with the sum of the coordinates equal to A_n. On the other hand, in view of Lemma 5.1 the nucleolus z is an element in M, and $Z_n = A_n$ because z is a core element, so the efficiency condition holds. Hence $\{z\} = \{x \in M \mid X_n = Z_n\}$, which proves the theorem. ■

Remark 5.1. The result of Theorem 5.1 is used in Brânzei et al. (2000) to design an iterative algorithm in order to approximate the nucleolus.

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